THE STELLAR TRIANGULATION WITH PHOTOGRAPHIC OBSERVATIONS

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(Received October 10, 1964)

Abstract. A general method for the adjustment of stellar triangulation using photographic observations of artificial satellites, missiles or rockets with stars in the background is suggested. This approach recognizes the plate readings as basic data and makes use of the astronomical method of dependences in establishing the conditional equations. The method is valid for both stationary and nonstationary camera systems. Its simplicity makes possible the adjustment in a block – as organic unit – of a world-wide triangulation with photographic observations. Since it is intended to establish a working system from this theory, some controversial engineering aspects are discussed while the method is critically compared with the familiar photogrammetric approach. A historical review and a synthesis of the contemporary efforts for the development of a world-wide stellar triangulation are presented.

1. Introduction

The stellar triangulation with photographic observations fulfils simultaneously two immediate objectives: (1) the unprecedented accurate space location of artificial satellites, missiles and rockets and (2) the building of a globe-wide network of supercontrol points.

The precise location of space objects is important for problems of national defense and for the investigation of space phenomena; e.g., effects on space navigation due to gravity and magnetic anomalies, air drag, cosmic radiation, etc.

On the other hand, a network of supercontrol points distributed conveniently over continents and islands (Figure 1) to satisfy the stringent cartographic needs, would benefit every country.

A global network eliminates the existing discrepancies among the national triangulations and makes possible the reduction of all mapping systems to a unique datum.

Furthermore, the "geodetic integration" of the world throughout a stellar triangulation is helpful for the accurate determination of continental tides and for valid studies about the equilibrium of the earth's core.

The unprecedented accuracy of the stellar triangulation is owed to its photographic observations which (unlike the visual observations) are not affected by the systematic errors of the gravity field.

This feature is remarkable because it opens the way for the strictly geometric determination of the earth figure without reference to gravity and its anomalies.

The scientific and practical implications of these and other prospects are, indeed, farreaching. They already have been recognized by the XIII General Assembly of the International Union of Geodesy and Geophysics held in Berkeley, California,

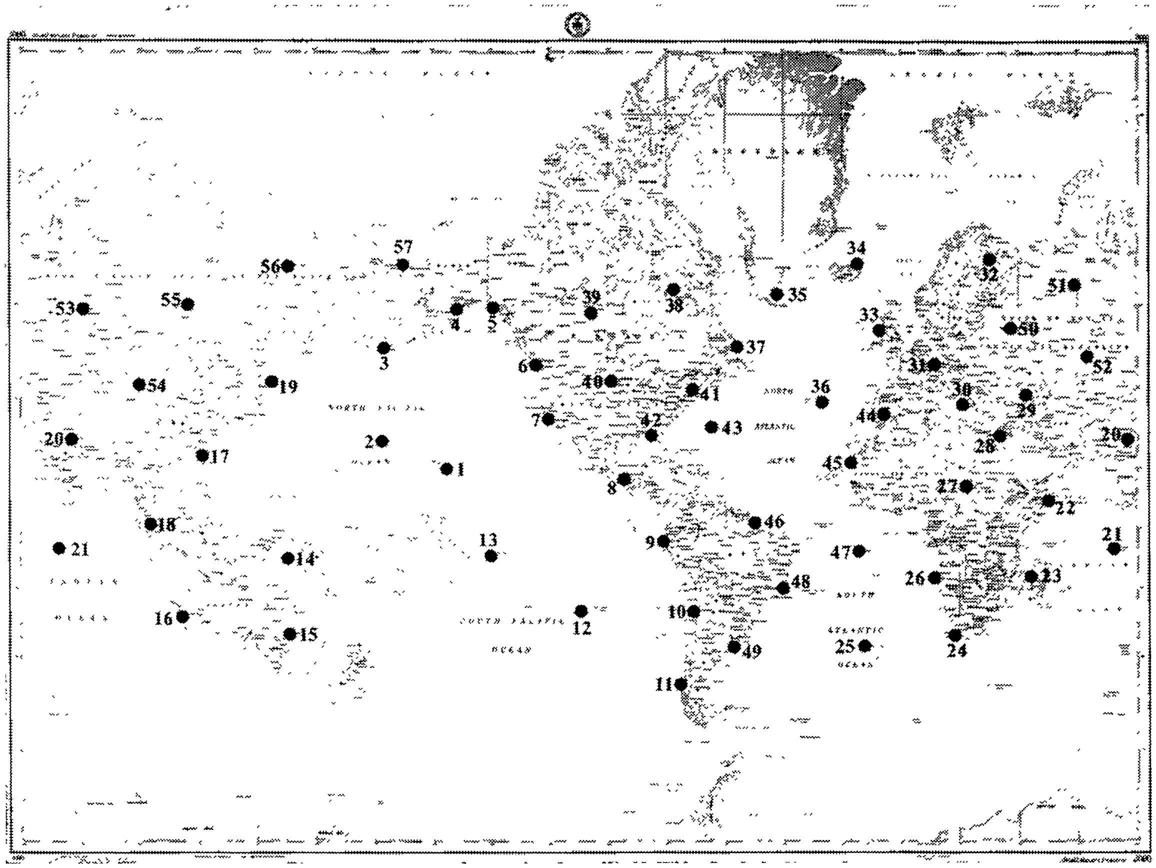

Fig. 1. Tentative suggestion for a world-wide geodetic network.

USA, August 1963, which adopted a resolution recommending immediate cooperation among nations for the effective prosecution of a world-wide stellar triangulation.

2. Background

It is apparent that stationary and nonstationary camera systems are used for the observation of the stellar triangulation.

A comprehensive discussion of both systems will be presented in this paper.

The discovery of the stellar triangulation as well as the first observations with a stationary camera are to be credited to Hoppman and Lohman (SCHMID, 1963). These German engineers used it in the 1930's for trajectory measurements of shells and small rockets. Then the method became part of the rocket testing in Peenemünde (Germany).

Because of the classified character of the work, no detail information of the first successful results is available.

The original idea by Hoppman and Lohman, however, was to photograph space flares against the starry sky, the same way as photographing terrestrial objects surrounded by geodetic points. Thus, the mathematics of the stellar triangulation

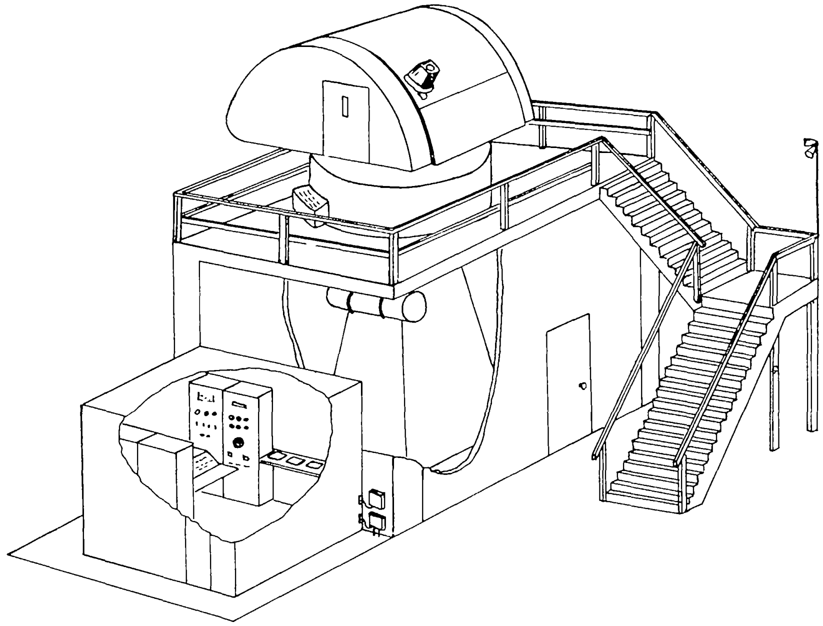

Fig. 2. The BC–4 ballistic camera installation at the Atlantic Missile Range, Air Force Missile Test Center, Patrick Air Force Base, Florida, U.S.A.

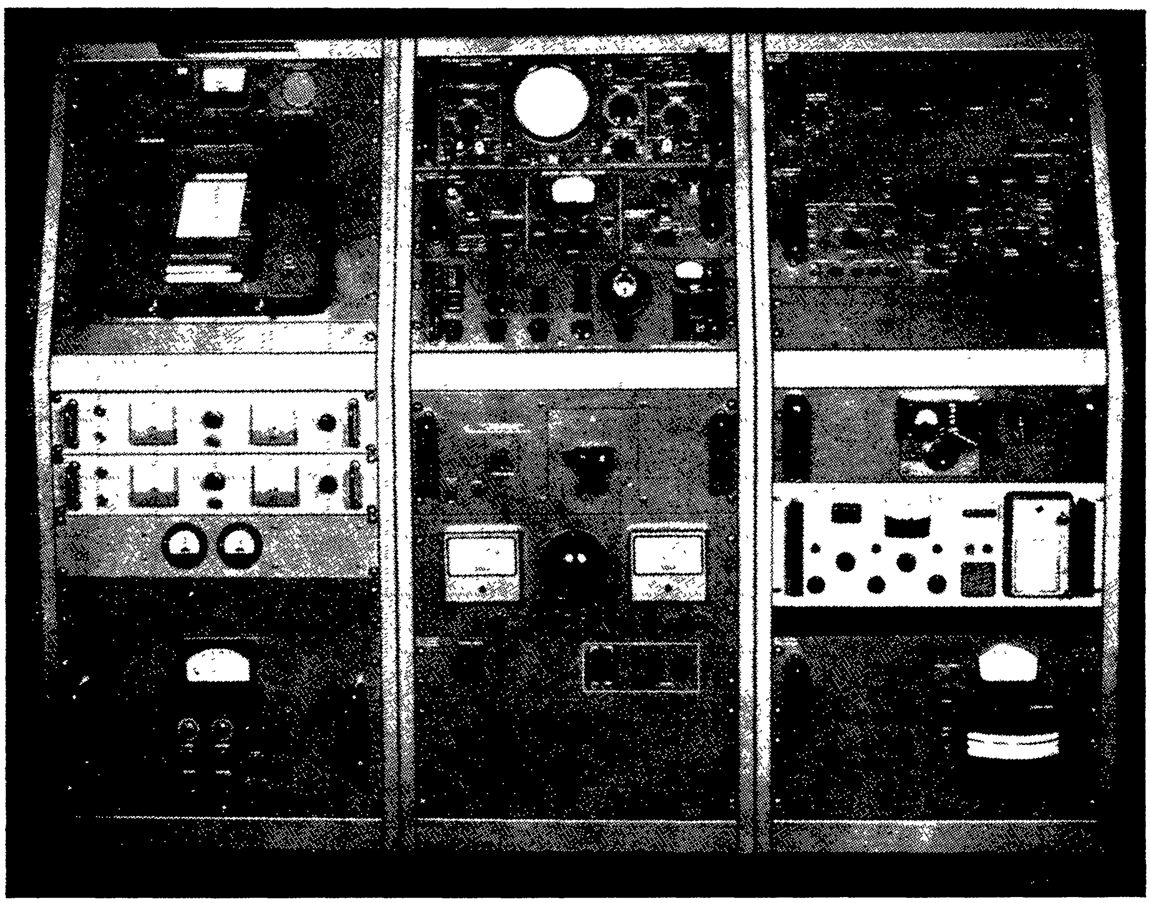

Fig. 3. High precision shutter drive mechanism used by the U.S. Coast and Geodetic Survey.

with a fixed camera may be found in standard manuals of photogrammetry. That is why the stationary camera system is usually known as a photogrammetric method.

After the Second World War, German immigrants brought the photogrammetric method in the United States.

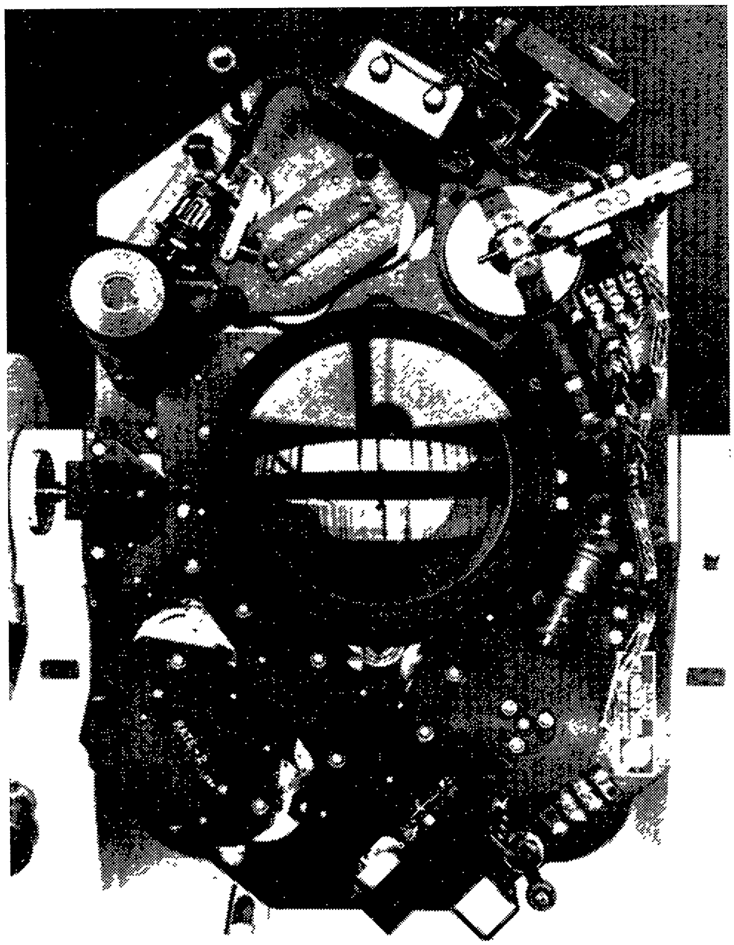

Fig. 4. High-precision gearing used by the U.S. Coast and Geodetic Survey for star and satellite trail chopping.

Initially, it was adopted by the Army Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland, and then by the Air Force Missile Test Center, Patrick Air Force Base, Florida (Fig. 2).

In 1962, U.S. Coast and Geodetic Survey also adopted the photogrammetric method for its new Satellite Geodesy Program.

On the other hand, the mobile camera system has been suggested by VÄISÄLÄ (1946). He was the first to take photographs (with an equatorially mounted and driver reflecting telescope designed by himself) of flares against the stars on clear nights when the flares were visible from distant camera stations.

Because of the postwar shortage in material, Väisälä's experiments had to be interrupted and only in 1959 could they start again in collaboration with the Geodetic Institute in Finland (KUKKAMÄKI, 1959).

On account of the quite satisfactory results of these field experiments, the Geodetic Institute in Finland decided upon the prosecution of a stellar triangulation across the country. For this purpose, three Väisälä reflecting telescopes have been ordered (at a price of $ 20 thousand each) and are presently under construction (KUKKAMÄKI, 1963).

Meanwhile, quite refined and expensive experiments with the stationary camera were conducted by various United States agencies. As an example, the actual photogrammetric system of U. S. Coast and Geodetic Survey is a matter of 8 years of development at an estimated cost of some $ 3 million (TAYLOR and LAMPTON, 1964).

The system was designed around the Wild BC-4 (300 mm focus) ballistic camera for a network of satellite triangulation over the United States (YEAGER, 1964). Apparently, the experiments by U. S. Coast and Geodetic Survey led to the development of two accessories; an electronic synchronization device (Figure 3), and a high-precision gearing (Figures 4). The synchronization device operates the shutter timing and synchronization in a multistation observing program within ± 100 microseconds, while the high-precision gearing is responsible for the passive satellite and star trail chopping (U. S. Coast and Geodetic Survey, 1962).

This report suggests a new stellar triangulation theory valid for both fixed and mobile camera systems. Earlier results of the method have been presented at the Forty-First and Forty-Third Annual Meetings of the American Geophysical Union (CORPACIUS 1960, 1962) and at the XIII General Assembly of the International Union of Geodesy and Geophysics, 1963.

3. Description of the Theory

Starting with the improved Method of Dependences (COMRIE, 1929), assume a set of simultaneous photographs of a flare with stars in the background made from various camera stations.

The direction cosines of the rays joining the flare and the camera stations can be accurately computed by the following procedure:

Assume three material points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ forming a triangle. The obvious barycentric coordinates (x_0, y_0) of any point P_0 which is coplanar with $P_1P_2P_3$ can be expressed by

$$x_0 = D_1x_1 + D_2x_2 + D_2x_3 \tag{1}$$

$$y_0 = D_1y_1 + D_2y_2 + D_3y_3 \tag{2}$$

where the arbitrary parameters D_1, D_2, D_3, called dependences, are the mass of the points P_1, P_2, P_3, respectively.

Now determine the parameters $D_i (i=1, 2, 3)$ by satisfying the condition

$$1 = D_1 + D_2 + D_3 \tag{3}$$

Thus, the solution of Equations (1), (2) and (3) leads to the unique value of the parameters

$$D_1 = \frac{\Delta_1}{\Delta}; \qquad D_2 = \frac{\Delta_2}{\Delta}; \qquad D_3 = \frac{\Delta_3}{\Delta} \tag{4}$$

Where

$$\Delta_1 = x_0 y_2 - x_2 y_0 + x_3 y_0 - x_0 y_3 + x_2 y_3 - x_3 y_2 \tag{5}$$

$$\Delta_2 = x_0 y_3 - x_3 y_0 + x_1 y_0 - x_0 y_1 + x_3 y_1 - x_1 y_3 \tag{6}$$

$$\Delta_3 = x_0 y_1 - x_1 y_0 + x_2 y_0 - x_0 y_2 + x_1 y_2 - x_2 y_1 \tag{7}$$

$$\Delta = x_1 y_2 - x_2 y_1 + x_3 y_1 - x_1 y_3 + x_2 y_3 - x_3 y_2 . \tag{8}$$

If more than three stars are available, the least squares solution leads to adjusted $D_i (i = 1, 2, \ldots, n)$ given by the formula (4′):

$$D_i = B(x_i - \bar{x}) + C(y_i - \bar{y}) + \frac{1}{n} \tag{4′}$$

where

$$A = [(x - \bar{x})^2][(y - \bar{y})^2] - [(x - \bar{x})(y - \bar{y})]^2 \tag{5′}$$

$$B = \frac{1}{A}\{(x_0 - \bar{x})[(y - \bar{y})^2] - (y_0 - \bar{y})[(x - \bar{x})(y - \bar{y})]\} \tag{6′}$$

$$C = \frac{1}{A}\{(y_0 - \bar{y})[(x - \bar{x})^2] - (x_0 - \bar{x})[(x - \bar{x})(y - \bar{y})]\} \tag{7′}$$

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}; \qquad \bar{y} = \frac{\sum_{i=1}^{n} y_i}{n} \tag{8′}$$

where n is the number of stars. See also GENARO and TAFFARA (1946).

Since more than three stars do not improve the final result (COMRIE, 1929), the treatment to follow is referred to only three stars.

It is important to emphasize that Equations (1) and (2) conserve their form by every linear transformation of x, y. Thus, as consequence of condition (3), D_i are invariants against linear transformations of x_i, y_i and x_0, y_0, which logically show that translation and orientation of axes of coordinates and scale factor are immaterial so far as D_i is concerned. From this fact arises the astronomical method of dependences for computing the right ascension α and declination δ of stars, without knowing orientation calibration of camera and scale factor of photographs.

The formulae by COMRIE (1929):

$$\alpha_0 = \bar{\alpha} + \sum D_i(\alpha_i - \bar{\alpha}) - \tan\bar{\delta}\left[\sum D_i(\alpha_i - \bar{\alpha})(\delta_i - \bar{\delta}) - (\alpha_0 - \bar{\alpha})(\delta_0 - \bar{\delta})\right] \tag{9}$$

$$\delta_0 = \bar{\delta} + \sum D_i(\delta_i - \bar{\delta}) + \frac{\sin 2\bar{\delta}}{4}\left[\sum D_i(\alpha_i - \alpha)^2 - (\alpha_0 - \bar{\alpha})^2\right] \tag{10}$$

$$(i = 1, 2, 3)$$

might be used for computing the spherical coordinates α_0, δ_0 of a new star by means of dependences D_i and spherical coordinates $\alpha_i \delta_i$ of three reference stars. ($\bar{\alpha}$, $\bar{\delta}$ are the spherical coordinates of an arbitrary reference center. For computational simplicity, the reference center can be chosen in one of the above reference stars.)

Returning to the stellar triangulation, an absolute (sidereal) coordinate system $O\bar{X}\bar{Y}\bar{Z}$ (Figure 5) considered in the center of a reference ellipsoid is adopted. (The axis $O\bar{X}$ coincides with the line of equinox, $O\bar{Z}$ with the line of poles and $O\bar{Y}$ is perpendicular on the plane $\bar{X}O\bar{Z}$ and directed toward the east.)

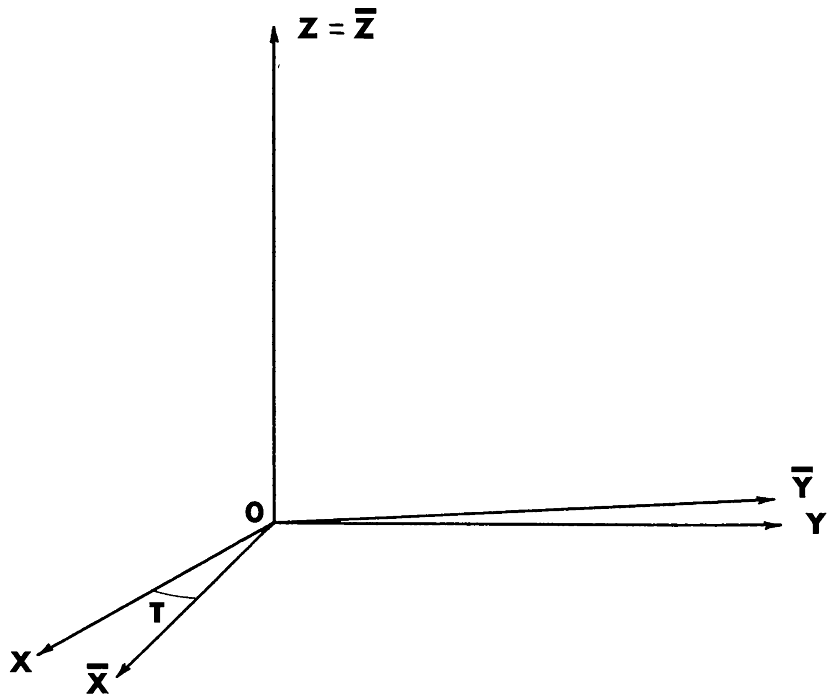

Fig. 5. The astronomic (absolute) and the terrestrial coordinate systems.

The direction cosines

$$u = \cos\alpha_0 \cos\delta_0 \tag{11}$$

$$v = \sin\alpha_0 \cos\delta_0 \tag{12}$$

$$w = \sin\delta_0 \tag{13}$$

of the ray joining the camera station and flare location between stars, can be computed by means of Formulae (9) and (10). Moreover, camera station $C(\bar{X}_C, \bar{Y}_C, \bar{Z}_C)$ and flare position $P_0(\bar{X}, \bar{Y}, \bar{Z})$ are related by formulae

$$\frac{\bar{Y} - \bar{Y}_C}{\bar{X} - \bar{X}_C} = \frac{v}{u} = \tan\alpha_0 \tag{14}$$

$$\frac{\bar{Z} - \bar{Z}_C}{\bar{X} - \bar{X}_C} = \frac{w}{u} = \frac{\tan\delta_0}{\cos\alpha_0} \tag{15}$$

or

$$\bar{\Phi} = (\bar{X} - \bar{X}_C)\sin\alpha_0 - (\bar{Y} - \bar{Y}_C)\cos\alpha_0 = 0 \tag{16}$$

$$\bar{\Psi} = (\bar{X} - \bar{X}_C)\sin\alpha_0 - (\bar{Z} - \bar{Z}_C)\cos\alpha_0\cos\delta_0 = 0\,. \tag{17}$$

Since the stellar triangulation is a geodetic problem, it is convenient to express the formulae (16) and (17) in the terrestrial space-time system $OXYZT$ (Figure 5).

The transformation from the sidereal system $O\bar{X}\bar{Y}\bar{Z}$ (which does not rotate with the earth) to the terrestrial or geodetic space-time system is given by the formulae:

$$\begin{bmatrix} \bar{X} \\ \bar{Y} \\ \bar{Z} \end{bmatrix} = \begin{bmatrix} \cos T & \sin T & 0 \\ -\sin T & \cos T & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{18}$$

where T is the geodetic time of the event.

The idea of geodetic time is introduced to avoid confusion with the standard time.

As a result of (18), Equations (16) and (17) become

$$\Phi = (X - X_C)\sin(\alpha_0 - T) - (Y - Y_C)\cos(\alpha_0 - T) = 0 \tag{19}$$

and

$$\Psi = (X - X_C)\sin\delta_0 - (Z - Z_C)\cos(\alpha_0 - T)\cos\delta_0 = 0\,. \tag{20}$$

The formulae (19) and (20) enjoy two important properties: they are general and independent. In other words, any projective relationship between a camera station and flare might be somehow related to (19) and (20).

It is suitable to convert (19) and (20) into conditional euqations by expressing α_0 and δ_0 in function of independently measured elements which are the plate co-ordinates $x_0 y_0$ of the flare P_0 and $x_i y_i$ of stars $P_i (i=1, 2, 3)$.

Therefore, after replacing in the last terms of (9) and (10) α_0—$\bar{\alpha}$ by $\Sigma(\alpha_i—\bar{\alpha})D_i$ and δ_0—$\bar{\delta}$ by $\Sigma(\delta_i—\bar{\delta})D_i$, Equations (9) and (10) are written again in proper manner:

$$\begin{aligned} \alpha_0 = \bar{\alpha} &+ [1 - (\delta_1 - \bar{\delta})\tan\bar{\delta}](\alpha_1 - \bar{\alpha})D_1 + [1 - (\delta_2 - \bar{\delta})\tan\bar{\delta}](\alpha_2 - \bar{\alpha})D_2 \\ &+ [1 - (\delta_3 - \bar{\delta})\tan\bar{\delta}](\alpha_3 - \bar{\alpha})D_3 + (\alpha_1 - \bar{\alpha})(\delta_1 - \bar{\delta})\tan\bar{\delta}\cdot D_1^2 \\ &+ (\alpha_2 - \bar{\alpha})(\delta_2 - \bar{\delta})\tan\bar{\delta}\cdot D_2^2 + (\alpha_3 - \bar{\alpha})(\delta_3 - \bar{\delta})\tan\bar{\delta}\cdot D_3^2 \\ &+ \tan\bar{\delta}[(\alpha_1 - \bar{\alpha})(\delta_2 - \bar{\delta}) + (\alpha_2 - \bar{\alpha})(\delta_1 - \bar{\delta})]D_1D_2 \\ &+ \tan\bar{\delta}[(\alpha_2 - \bar{\alpha})(\delta_3 - \bar{\delta}) + (\alpha_3 - \bar{\alpha})(\delta_2 - \bar{\delta})]D_2D_3 \\ &+ \tan\bar{\delta}[(\alpha_3 - \bar{\alpha})(\delta_1 - \bar{\delta}) + (\alpha_1 - \bar{\alpha})(\delta_3 - \bar{\delta})]D_3D_1 \end{aligned} \tag{21}$$

$$\begin{aligned} \delta_0 = \bar{\delta} &+ \left[(\delta_1 - \bar{\delta}) + (\alpha_1 - \bar{\alpha})^2\frac{\sin 2\bar{\delta}}{4}\right]D_1 \\ &+ \left[(\delta_2 - \bar{\delta}) + (\alpha_2 - \bar{\alpha})^2\frac{\sin 2\bar{\delta}}{4}\right]D_2 + \left[(\delta_3 - \bar{\delta}) + (\alpha_3 - \bar{\alpha})^2\frac{\sin 2\bar{\delta}}{4}\right]D_3 \\ &- (\alpha_1 - \bar{\alpha})^2\frac{\sin 2\bar{\delta}}{4}D_1^2 - (\alpha_2 - \bar{\alpha})^2\frac{\sin 2\bar{\delta}}{4}D_2^2 \\ &- (\alpha_3 - \bar{\alpha})^2\frac{\sin 2\bar{\delta}}{4}D_3^2 - (\alpha_1 - \bar{\alpha})(\alpha_2 - \bar{\alpha})\frac{\sin 2\bar{\delta}}{2}D_1D_2 \\ &- (\alpha_2 - \bar{\alpha})(\alpha_3 - \bar{\alpha})\frac{\sin 2\bar{\delta}}{2}D_2D_3 - (\alpha_3 - \bar{\alpha})(\alpha_1 - \bar{\alpha})\frac{\sin 2\bar{\delta}}{2}D_3D_1 \end{aligned} \tag{22}$$

After introducing the auxiliaries:

$$
\begin{aligned}
A_1 &= [1-(\delta_1-\bar\delta)\tan\bar\delta](\alpha_1-\bar\alpha); & A_1' &= (\alpha_1-\bar\alpha)(\delta_1-\bar\delta)\tan\bar\delta \\
A_2 &= [1-(\delta_2-\bar\delta)\tan\bar\delta](\alpha_2-\bar\alpha); & A_2' &= (\alpha_2-\bar\alpha)(\delta_2-\bar\delta)\tan\bar\delta \\
A_3 &= [1-(\delta_3-\bar\delta)\tan\bar\delta](\alpha_3-\bar\alpha); & A_3' &= (\alpha_3-\bar\alpha)(\delta_3-\bar\delta)\tan\bar\delta
\end{aligned}
$$

$$
\begin{aligned}
A_{12} &= \tan\bar\delta[(\alpha_1-\bar\alpha)(\delta_2-\bar\delta)+(\alpha_2-\bar\alpha)(\delta_1-\bar\delta)] \\
A_{23} &= \tan\bar\delta[(\alpha_2-\bar\alpha)(\delta_3-\bar\delta)+(\alpha_3-\bar\alpha)(\delta_2-\bar\delta)] \\
A_{31} &= \tan\bar\delta[(\alpha_3-\bar\alpha)(\delta_1-\bar\delta)+(\alpha_1-\bar\alpha)(\delta_3-\bar\delta)]
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
B_1 &= (\delta_1-\bar\delta)+(\alpha_1-\bar\alpha)^2\frac{\sin 2\bar\delta}{4}; & B_1' &= -(\alpha_1-\bar\alpha)^2\frac{\sin 2\bar\delta}{4} \\
B_2 &= (\delta_2-\bar\delta)+(\alpha_2-\bar\alpha)^2\frac{\sin 2\bar\delta}{4}; & B_2' &= -(\alpha_2-\bar\alpha)^2\frac{\sin 2\bar\delta}{4} \\
B_3 &= (\delta_3-\bar\delta)+(\alpha_3-\bar\alpha)^2\frac{\sin 2\bar\delta}{4}; & B_3' &= -(\alpha_3-\bar\alpha)^2\frac{\sin 2\bar\delta}{4}
\end{aligned}
$$

$$
\begin{aligned}
B_{12} &= -(\alpha_1-\bar\alpha)(\alpha_2-\bar\alpha)\frac{\sin 2\bar\delta}{2} \\
B_{23} &= -(\alpha_2-\bar\alpha)(\alpha_3-\bar\alpha)\frac{\sin 2\bar\delta}{2} \\
B_{31} &= -(\alpha_3-\bar\alpha)(\alpha_1-\bar\alpha)\frac{\sin 2\bar\delta}{2}
\end{aligned}
$$

Expressions (21) and (22) easily become:

$$
\alpha_0 = \bar\alpha + A_1D_1 + A_2D_2 + A_3D_3 + A_1'D_1^2 + A_2'D_2^2 + A_3'D_3^2 + A_{12}D_1D_2 + A_{23}D_2D_3 + A_{31}D_3D_1 \tag{24}
$$

$$
\delta_0 = \bar\delta + B_1D_1 + B_2D_2 + B_3D_3 + B_1'D_1^2 + B_2'D_2^2 + B_3'D_3^2 + B_{13}D_1D_2 + B_{23}D_2D_3 + B_{31}D_3D_1 \tag{25}
$$

Note that dependences D_1, D_2, D_3 are functions of directly measured plate coordinates x_0, y_0 and $x_i, y_i (i=1, 2, 3)$ because of relationships expressed in (4), (5), (6), (7) and (8). This solves the problem where the formulae (19) and (20) become conditional equations.

The Taylor expansion of (19) and (20) leads to the following linearized form:

$$
\Phi = \Phi_0 + \left(\frac{\partial\Phi}{\partial\alpha_0}\right)_0 \mathrm{d}\alpha_0 + \left(\frac{\partial\Phi}{\partial X_C}\right)_0 \mathrm{d}X_C + \left(\frac{\partial\Phi}{\partial Y_C}\right)_0 \mathrm{d}Y_C + \left(\frac{\partial\Phi}{\partial X}\right)_0 \mathrm{d}X + \left(\frac{\partial\Phi}{\partial Y}\right)_0 \mathrm{d}Y + \left(\frac{\partial\Phi}{\partial T}\right)_0 \mathrm{d}T = 0 \tag{26}
$$

$$
\Psi + \Psi_0 + \left(\frac{\partial\Psi}{\partial\alpha_0}\right)_0 \mathrm{d}\alpha_0 + \left(\frac{\partial\Psi}{\partial\delta_0}\right)_0 \mathrm{d}\delta_0 + \left(\frac{\partial\Psi}{\partial X_C}\right)_0 \mathrm{d}X_C + \left(\frac{\partial\Psi}{\partial Z_C}\right)_0 \mathrm{d}Z_C + \left(\frac{\partial\Psi}{\partial X}\right)_0 \mathrm{d}X + \left(\frac{\partial\Psi}{\partial Z}\right)_0 \mathrm{d}Z + \left(\frac{\partial\Psi}{\partial T}\right)_0 \mathrm{d}T = 0 \tag{27}
$$

where $d\alpha_0$, $d\delta_0$ may be eliminated by the help of (24) and (25) and supplemented by (4) through (8).

The differential of (24) and (25) results in:

$$d\alpha_0 = A\,dD_1 + B\,dD_2 + D\,dD_3 \tag{28}$$

$$d\delta_0 = E\,dD_1 + F\,dD_2 + G\,dD_3 \tag{29}$$

where A, B, C, E, F, G and $dD_i (i=1, 2, 3)$ are given by:

$$\begin{aligned} A &= A_1 + 2A_1' D_1 + A_{12} D_2 + A_{31} D_3; & E &= B_1 + 2B_1' D_1 + B_{12} D_2 + B_{31} D_3 \\ B &= A_2 + 2A_2' D_2 + A_{23} D_3 + A_{12} D_1; & F &= B_2 + 2B_2' D_2 + B_{23} D_3 + B_{12} D_1 \\ C &= A_3 + 2A_3' D_3 + A_{31} D_1 + A_{23} D_2; & G &= B_3 + 2B_3' D_3 + B_{31} D_1 + B_{23} D_2 \end{aligned} \tag{30}$$

and

$$dD_i = \frac{\partial D_i}{\partial x_0} v_{x_0} + \frac{\partial D_i}{\partial y_0} v_{y_0} + \frac{\partial D_i}{\partial x_1} v_{x_1} + \frac{\partial D_i}{\partial y_1} v_{y_1} + \frac{\partial D_i}{\partial x_2} v_{x_2} + \frac{\partial D_i}{\partial y_2} v_{y_2} + \frac{\partial D_i}{\partial x_3} v_{x_3} + \frac{\partial D_i}{\partial y_3} v_{y_3}$$

$$(i = 1, 2, 3)$$

Here v_{x_0}, v_{y_0}, ..., etc., are unknown discrepancies (increments) corresponding to the plate measurements $x_0, y_0, \ldots$, etc., respectively, while $\partial D_i/\partial x_0$, $\partial D_i/\partial y_0$, ..., etc. may be computed from (4), (5), ... (8).

We have

$$\begin{aligned} \frac{\partial D_1}{\partial x_0} &= \frac{-(y_3 - y_2)}{\Delta}; & \frac{\partial D_1}{\partial y_0} &= \frac{(x_3 - x_2)}{\Delta} \\ \frac{\partial D_1}{\partial x_1} &= \frac{\Delta_1 (y_3 - y_2)}{\Delta^2}; & \frac{\partial D_1}{\partial y_1} &= \frac{-\Delta_1 (x_3 - x_2)}{\Delta^2} \\ \frac{\partial D_1}{\partial x_2} &= \frac{\Delta (y_3 - y_0) - \Delta_1 (y_3 - y_1)}{\Delta^2}; & \frac{\partial D_1}{\partial y_2} &= \frac{-\Delta (x_3 - x_0) + \Delta_1 (x_3 - x_1)}{\Delta^2} \\ \frac{\partial D_1}{\partial x_3} &= \frac{-\Delta (y_2 - y_0) + \Delta_1 (y_2 - y_1)}{\Delta^2}; & \frac{\partial D_1}{\partial y_3} &= \frac{\Delta (x_2 - x_0) - \Delta_1 (x_2 - x_1)}{\Delta^2} \end{aligned} \tag{31}$$

$$\begin{aligned} \frac{\partial D_2}{\partial x_0} &= \frac{y_3 - y_1}{\Delta}; & \frac{\partial D_2}{\partial y_0} &= \frac{-(x_3 - x_1)}{\Delta} \\ \frac{\partial D_2}{\partial x_1} &= \frac{-\Delta (y_3 - y_0) + \Delta_2 (y_3 - y_2)}{\Delta^2}; & \frac{\partial D_2}{\partial y_1} &= \frac{\Delta (x_3 - x_0) - \Delta_2 (x_3 - x_2)}{\Delta^2} \\ \frac{\partial D_2}{\partial x_2} &= \frac{-\Delta_2 (y_3 - y_1)}{\Delta^2}; & \frac{\partial D_2}{\partial y_2} &= \frac{\Delta_2 (x_3 - x_1)}{\Delta^2} \\ \frac{\partial D_2}{\partial x_3} &= \frac{\Delta (y_1 - y_0) + \Delta_2 (y_2 - y_1)}{\Delta^2}; & \frac{\partial D_2}{\partial y_3} &= \frac{-\Delta (x_1 - x_0) - \Delta_2 (x_2 - x_1)}{\Delta^2} \end{aligned} \tag{32}$$

$$\frac{\partial D_3}{\partial x_0} = \frac{-(y_2 - y_1)}{\Delta}; \qquad \frac{\partial D_3}{\partial y_0} = \frac{(x_2 - x_1)}{\Delta}$$
$$\frac{\partial D_3}{\partial x_1} = \frac{\Delta(y_2 - y_0) + \Delta_3(y_3 - y_2)}{\Delta^2}; \qquad \frac{\partial D_3}{\partial y_1} = \frac{-\Delta(x_2 - x_0) - \Delta_3(x_3 - x_2)}{\Delta^2}$$
$$\frac{\partial D_3}{\partial x_2} = \frac{-\Delta(y_1 - y_0) - \Delta_3(y_3 - y_1)}{\Delta^2}; \qquad \frac{\partial D_3}{\partial y_2} = \frac{\Delta(x_1 - x_0) + \Delta_3(x_3 - x_1)}{\Delta^2} \qquad (33)$$
$$\frac{\partial D_3}{\partial x_3} = \frac{\Delta_3(y_2 - y_1)}{\Delta^2}; \qquad \frac{\partial D_3}{\partial y_3} = \frac{-\Delta_3(x_2 - x_1)}{\Delta^2}$$

Finally, the absolute terms Φ_0, Ψ_0, in (26) and (27) and the coefficients

$$\left(\frac{\partial \Phi}{\partial \alpha_0}\right)_0, \quad \left(\frac{\partial \Phi}{\partial X_C}\right)_0 \ldots \text{etc.}$$

and

$$\left(\frac{\partial \Psi}{\partial \alpha_0}\right)_0, \quad \left(\frac{\partial \Psi}{\partial \delta_0}\right)_0, \quad \left(\frac{\partial \Psi}{\partial X_C}\right)_0 \ldots \text{etc.}$$

are readily apparent from (19) and (20):

$$\Phi_0 = \varepsilon = [(X - X_C)\sin(\alpha_0 - T) - (Y - Y_C)\cos(\alpha_0 - T)]_0$$
$$\Psi_0 = \varepsilon' = [(X - X_C)\sin\delta_0 - (Z - Z_C)\cos(\alpha_0 - T)\cos\delta_0]_0$$
$$\left(\frac{\partial \Phi}{\partial \alpha_0}\right)_0 = [(X - X_C)\cos(\alpha_0 - T) + (Y - Y_C)\sin(\alpha_0 - T)]_0$$
$$\left(\frac{\partial \Psi}{\partial \alpha_0}\right)_0 = [(Z - Z_C)\sin(\alpha_0 - T)\cos\delta_0]_0$$
$$\left(\frac{\partial \Psi}{\partial \delta_0}\right)_0 = [(X - X_C)\cos\delta_0 + (Z - Z_C)\cos(\alpha_0 - T)\sin\delta_0]_0$$
$$\left(\frac{\partial \Phi}{\partial X_C}\right)_0 = U_C = [-\sin(\alpha_0 - T)]_0; \qquad \left(\frac{\partial \Psi}{\partial X_C}\right)_0 = U'_C = [-\sin\delta_0]_0 \qquad (34)$$
$$\left(\frac{\partial \Phi}{\partial Y_C}\right)_0 = V_C = [\cos(\alpha_0 - T)]_0; \qquad \left(\frac{\partial \Psi}{\partial Z_C}\right)_0 = W'_C = [\cos(\alpha_0 - T)\cos\delta_0]_0$$
$$\left(\frac{\partial \Phi}{\partial X}\right)_0 = U = [\sin(\alpha_0 - T)]_0; \qquad \left(\frac{\partial \Psi}{\partial X}\right)_0 = U' = [\sin\delta_0]_0$$
$$\left(\frac{\partial \Phi}{\partial Y}\right)_0 = V = [-\cos(\alpha_0 - T)]_0; \qquad \left(\frac{\partial \Psi}{\partial Z}\right)_0 = W' = [-\cos(\alpha_0 - T)\cos\delta_0]_0$$
$$\left(\frac{\partial \Phi}{\partial T}\right)_0 = P = [-(X - X_C)\cos(\alpha_0 - T) - (Y - Y_C)\sin(\alpha_0 - T)]_0$$
$$\left(\frac{\partial \Psi}{\partial T}\right)_0 = P' = [-(Z - Z_C)\sin(\alpha_0 - T)\cos\delta_0]_0 .$$

All auxiliaries being determined, the expressions

$$\left(\frac{\partial \Phi}{\partial \alpha_0}\right)_0 \mathrm{d}\alpha_0, \quad \left(\frac{\partial \Psi}{\partial \alpha_0}\right)_0 \mathrm{d}\alpha_0 \quad \text{and} \quad \left(\frac{\partial \Psi}{\partial \delta_0}\right)_0 \mathrm{d}\delta_0$$

which enter in the linearized conditional equations (26) and (27) may be expressed in terms of physical elements.

Referring to (28), (30), (31), (32), (33), and (34), we have:

$$\left(\frac{\partial\Phi}{\partial\alpha_0}\right)_0 \mathrm{d}\alpha_0 = a_0 v_{x_0} + b_0 v_{y_0} + a_1 v_{x_1} + b_1 v_{y_1} + a_2 v_{x_2} + b_2 v_{y_2} + a_3 v_{x_3} + b_3 v_{y_3} \tag{35}$$

where

$$\begin{aligned}
a_0 &= \left(\frac{\partial\Phi}{\partial\alpha_0}\right)_0 \frac{1}{\Delta}\left[-A(y_3-y_2)+B(y_3-y_1)-C(y_2-y_1)\right]\\
b_0 &= \left(\frac{\partial\Phi}{\partial\alpha_0}\right)_0 \frac{1}{\Delta}\left[+A(x_3-x_2)-B(x_3-x_1)+C(x_2-x_1)\right]\\
a_1 &= \left(\frac{\partial\Phi}{\partial\alpha_0}\right)_0 \frac{1}{\Delta^2}\left[+A\Delta_1(y_3-y_2)-B\Delta(y_3-y_0)+B\Delta_2(y_3-y_2)\right.\\
&\qquad\left.+C\Delta(y_2-y_0)+C\Delta_3(y_3-y_2)\right]\\
b_1 &= \left(\frac{\partial\Phi}{\partial\alpha_0}\right)_0 \frac{1}{\Delta^2}\left[-A\Delta_1(x_3-x_2)+B\Delta(x_3-x_0)-B\Delta_2(x_3-x_2)\right.\\
&\qquad\left.-C\Delta(x_2-x_0)-C\Delta_3(x_3-x_2)\right]\\
a_2 &= \left(\frac{\partial\Phi}{\partial\alpha_0}\right)_0 \frac{1}{\Delta^2}\left[+A\Delta(y_3-y_0)-A\Delta_1(y_3-y_1)-B\Delta_2(y_3-y_1)\right.\\
&\qquad\left.-C\Delta(y_1-y_0)-C\Delta_3(y_3-y_1)\right]\\
b_2 &= \left(\frac{\partial\Phi}{\partial\alpha_0}\right)_0 \frac{1}{\Delta^2}\left[-A\Delta(x_3-x_0)+A\Delta_1(x_3-x_1)+B\Delta_2(x_3-x_1)\right.\\
&\qquad\left.+C\Delta(x_1-x_0)+C\Delta_3(x_3-x_1)\right]\\
a_3 &= \left(\frac{\partial\Phi}{\partial\alpha_0}\right)_0 \frac{1}{\Delta^2}\left[-A\Delta(y_2-y_0)+A\Delta_1(y_2-y_1)+B\Delta(y_1-y_0)\right.\\
&\qquad\left.+B\Delta_2(y_2-y_1)+C\Delta_3(y_2-y_1)\right]\\
b_3 &= \left(\frac{\partial\Phi}{\partial\alpha_0}\right)_0 \frac{1}{\Delta^2}\left[+A\Delta(x_2-x_0)-A\Delta_1(x_2-x_1)-B\Delta(x_1-x_0)\right.\\
&\qquad\left.-B\Delta_2(x_2-x_1)-C\Delta_3(x_2-x_1)\right]
\end{aligned} \tag{36}$$

The same procedure as in (35) and (36) leads to:

$$\left(\frac{\partial\Psi}{\partial\alpha_0}\right)_0 \mathrm{d}\alpha_0 + \left(\frac{\partial\Psi}{\partial\delta_0}\right)_0 \mathrm{d}\delta_0 = a_0' v_{x_0} + b_0' v_{y_0} + a_1' v_{x_1} + b_1' v_{y_1} + a_2' v_{x_2} + b_2' v_{y_2} + a_3' v_{x_3} + b_3' v_{y_3} \tag{37}$$

where again

$$\begin{aligned}
a_0' &= \frac{1}{\Delta}\left[-L(y_3-y_2)+M(y_3-y_1)-N(y_2-y_1)\right]\\
b_0' &= \frac{1}{\Delta}\left[L(x_3-x_2)-M(x_3-x_1)+N(x_2-x_1)\right]\\
a_1' &= \frac{1}{\Delta^2}\left[L\Delta_1(y_3-y_2)-M\Delta(y_3-y_0)+M\Delta_2(y_3-y_2)\right.\\
&\qquad\left.+N\Delta(y_2-y_0)+N\Delta_3(y_3-y_2)\right]
\end{aligned} \tag{38}$$

$$
\begin{aligned}
b_1' &= \frac{1}{\Delta^2}[-L\Delta_1(x_3 - x_2) + M\Delta(x_3 - x_0) - M\Delta_2(x_3 - x_2) \\
&\qquad - N\Delta(x_2 - x_0) - N\Delta_3(x_3 - x_2)] \\
a_2' &= \frac{1}{\Delta^2}[L\Delta(y_3 - y_0) - L\Delta_1(y_3 - y_1) - M\Delta_2(y_3 - y_1) \\
&\qquad - N\Delta(y_1 - y_0) - N\Delta_3(y_3 - y_1)] \\
b_2' &= \frac{1}{\Delta^2}[-L\Delta(x_3 - x_0) + L\Delta_1(x_3 - x_1) + M\Delta_2(x_3 - x_1) \\
&\qquad + N\Delta(x_1 - x_0) + N\Delta_3(x_3 - x_1)] \\
a_3' &= \frac{1}{\Delta^2}[-L\Delta(y_2 - y_0) + L\Delta_1(y_2 - y_1) + M\Delta(y_1 - y_0) \\
&\qquad + M\Delta_2(y_2 - y_1) + N\Delta_3(y_2 - y_1)] \\
b_3' &= \frac{1}{\Delta^2}[L\Delta(x_2 - x_0) - L\Delta_1(x_2 - x_1) - M\Delta(x_1 - x_0) \\
&\qquad - M\Delta_2(x_2 - x_1) - N\Delta_3(x_2 - x_1)]
\end{aligned}
\tag{38}
$$

with

$$
L = \left(\frac{\partial\Psi}{\partial\alpha_0}\right)_0 A + \left(\frac{\partial\Psi}{\partial\delta_0}\right)_0 E; \quad M = \left(\frac{\partial\Psi}{\partial\alpha_0}\right)_0 B + \left(\frac{\partial\Psi}{\partial\delta_0}\right)_0 F; \quad N = \left(\frac{\partial\Psi}{\partial\alpha_0}\right)_0 C + \left(\frac{\partial\Psi}{\partial\delta_0}\right)_0 G.
$$

The similarity and cyclical structure of coefficients (36) and (38), which moreover characterize on the whole auxiliaries of this paper, simplify their deduction and computation, and make possible a thorough checking.

Placing (35) and (37) in (26) and (27), and in view of (34), the conditional equations of the transformation into the geodetic space-time coordinate system are

$$
\begin{aligned}
&a_0 v_{x_0} + b_0 v_{y_0} + a_1 v_{x_1} + b_1 v_{y_1} + a_2 v_{x_2} + b_2 v_{y_2} + a_3 v_{x_3} + b_3 v_{y_3} \\
&\quad + U_C \mathrm{d}X_C + V_C \mathrm{d}Y_C \quad + U \mathrm{d}X + V \mathrm{d}Y \quad + P \mathrm{d}T + \varepsilon = 0
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
&a_0' v_{x_0} + b_0' v_{y_0} + a_1' v_{x_1} + b_1' v_{y_1} + a_2' v_{x_2} + b_2' v_{y_2} + a_3' v_{x_3} + b_3' v_{y_3} \\
&\quad + U_C' \mathrm{d}X_C \quad + W_C' \mathrm{d}Z_C + U' \mathrm{d}X \quad + W' \mathrm{d}Z + P' \mathrm{d}T + \varepsilon' = 0
\end{aligned}
\tag{40}
$$

The coefficients in Equation (39) and (40) result from (34) combined with (36) and (38).

The conditional equations (39) and (40) represent the most general case of adjustment, namely the conditional adjustment with unknowns. All other existing forms of adjustment may be considered as particular problems of this general case (HELMERT, 1907).

The conditional equations (39) and (40) show that the flare location is defined by four variables, three in space and one in time. In other words, a trajectory with photographic observations can be adjusted point by point in four dimensional space-time.

Starting with the equations (39) and (40), three different computing methods can be developed. The first method is the direct one, derived from the unchanged set of conditional equations (39) and (40); the second one is the correlative adjustment resulting after the elimination of the unknowns $\mathrm{d}X, \mathrm{d}Y, \mathrm{d}Z, \mathrm{d}T$; and finally the mediate adjustment which is apparent when the discrepancies $v_{x_0}, v_{y_0}, v_{x_1}$, etc., are eliminated.

Since the computation to be chosen depends only on practical convenience, the numerical application will show preference for one or the other method.

Here the general method is adopted.

The system of error equations deducted from (39) and (40) and written in matrix symbols is

$$Av + UX + E = 0 \tag{41}$$

while the least squares give the correlative and conditional equations

$$pv = A^T K \qquad \text{and} \qquad U^T K = 0 \tag{42}$$

Notice: the transpose of the matrices A and U in (41) are denoted in (42) by A^T and U^T.

Obviously, the normal equations become

$$\begin{bmatrix} N & U \\ U^T & 0 \end{bmatrix} \begin{bmatrix} K \\ X \end{bmatrix} + \begin{bmatrix} E \\ 0 \end{bmatrix} = 0 \tag{43}$$

The solution of (43) and the subsequent error analysis are routine matters of the electronic computation.

Nevertheless, the following remarks by WOLF (1962) should be of interest:

One could also conceive $\mathrm{d}\alpha_0$ and $\mathrm{d}\delta_0$ of Equations (26) and (27) as correlated observations (i.e., mutually dependent) and for example to solve the matter according to TIENSTRA as Standard Problem No. II (*Bulletin Géodésique* 1947, No. 6, page 311). Whether one then ascertains the necessary $Q_{\alpha\alpha}, Q_{\alpha\delta}, Q_{\delta\delta}$ from the peculiar adjustment with more than three stars or simply based on your Equations (9) and (10), it would be irrelevant for the computation with your Equations (26) and (27). Perhaps, it is possible to build the conditional equations by choosing the reference stars in such a manner that α_0 and δ_0 become orthogonal functions ($Q_{\alpha\delta}=0$). Thus, $\mathrm{d}\alpha_0$ and $\mathrm{d}\delta_0$ in (26) and (27) could then be handled as independent observations, of course with weights $1/Q_{\alpha\alpha}$, $1/Q_{\delta\delta}$, respectively. The numerical result must naturally be absolutely the same as from your adjustment, Wolf concluded.

These remarks will be considered in future investigations. The present report, however, is referred to the three star dependence method only. As already pointed out, a higher number of stars is not expected to increase the accuracy of the final result.

The experience shows that it is difficult to find a sufficient number of stars (close to the flare) that α_0 and δ_0 become orthogonal functions. Nevertheless, Wolf's suggestion may be feasible if artificial stars will be marked on the plates (BRUCKLACHER, 1961) in such a way that the orthogonality condition is satisfied. In this case, an additional star catalogue of the artificial stars, which could be incorporated in the computation routine, is needed.

4. Discussion and Error Analysis

The conditional equations (39) and (40) have been deducted from (16) and (17) upon expressing α_0 and δ_0 in terms of plate readings.

By establishing the equations (39) and (40), no use of camera orientation-calibration is made. Hence, each conditional equation has six unknowns less (i.e., three angles of exterior orientation and three elements of calibration). The computation is simplified while the reduction is free of the systematic errors related to camera orientation-calibration.

The decrease in number of the unknowns of each plate makes feasible the general adjustment of an extensive stellar triangulation around the Earth as an organic unit.

A further benefit derived from Equations (30) and (40) is the correct study of error propagation due to the adjustment in a block. The adjustment divided in groups not only leads to an objectionable analysis of the error propagation, but it also corrupts the final results because of the inadequate foundation of the probability theory. More about the rigorous adjustment versus the forced adjustment divided in groups will be commented on in Section 6.

The accuracy of the conditional equations (39) and (40) is reflected by Comrie's equations (9) and (10) which are valid up to the second approximation in $\Delta\alpha_0$ and $\Delta\delta_0$.

Astronomical refraction, shimmer (refraction anomaly) resulting from atmospheric turbulence, scintillation, lens and emulsion distortion, etc., are causes of disturbances of the projective colinearity condition in image-space, and of inconsistencies in the initial conditional equations (16) and (17). These may be partially eliminated by skilled observation methods and refined photographic material and partially by reductions applied as independent corrections to the plate coordinates. Of course, residuals of such systematic errors are inevitable in the observation equations; however, the linear expression of these residuals is immaterial so far as dependences D_i are concerned while the non-linear ones are carried throughout the adjustment by the coefficients of the observation equations. Since the remainder of the errors are of the second order, they do not affect the adjustment; only the concept of dependences D_i is modified in order to regard them as including the residual effects of the mentioned corrections.

It is to be emphasized that the reduction method presented here uses 3 to 6 stars close to the flare which are needed in both computation and control. The registration of sufficient stars is self-evident with the mobile (e.g., the equatorially mounted and driven) camera for which this reduction method is intended.

In this method, one needs not be concerned with the space intersection of rays connecting two or more camera stations with the flare (BROWN, 1957). The conditional equations (39) and (40) lead to both adjusted observations and flare space locations. The rays will intersect in a point as part of the unique least squares solution.

Some 30 error sources affecting the photogrammetric camera system and its reduction method were pointed out by BROWN (1957). Since the published error diagram is by no means exhaustive, many more unknown errors are particularly harmful in a photogrammetric camera characterized by a reduced focal length and an extensive field of view.

In spite of these obstacles, photogrammetry has achieved a respected rank in technology. This is true because the aforementioned errors have in photogrammetry no practical significance inasmuch as the final photogrammetric product, the map, always involves a scale factor (in amount of 10^{-3} to 10^{-6} or less), small enough to obviate most of the error effects.

The stellar triangulation with photographic observations does not enjoy scale factor and therefore, the study of error propagation becomes quite important.

The usual procedure for the accuracy determination of the photogrammetric camera system requires (1) the rigorous evaluation of all sources of random and systematic errors, and (2) the complete knowledge of the error distribution over an extensive camera field. This approach is difficult to carry out because of insufficient information about the behaviour of so many error sources.

It is possible sometimes to diminish or eliminate the error effects by skilled observation and reduction methods instead of detecting and computing them one by one. And this is what the method of dependences and the reduction presented here do. This method should be immune to many of the aforementioned errors if certain conditions peculiar to it as well as to the standard astrographic method are satisfied.

Let us discuss the features which should substantiate this point and define the accuracy of the reduction method under consideration.

In the astrographic method, the following linear equations,

$$X_i = ax_i + by_i + c \tag{44}$$

$$Y_i = \mathrm{d}x_i + ey_i + f$$

relate the standard coordinates X_i, $Y_i(i=1, 2, 3, \ldots, n, 0)$ of n reference stars and a flare, with the measured coordinates x_i, y_i. The coefficients a, b, c, d, e and f are called the plate constants. The least squares solution of the $2n$ equations resulting from n reference stars leads to the numerical values of the plate constants.

It is to be pointed out that Equations (44) are valid only for the reference stars of a restricted field. This is always accomplished in astrometry where the photographic refractor and the reflecting telescope never exceed a 3° field of view. Obviously, for stars distributed over quite a large field (33° to 76°) of a BC–4 ballistic camera the equations (44) are not at all valid. They have to be completed with second and higher order terms, and the errors of the field deformation must be kept under special control (TURNER, 1893).

In this case the number of the plate constants increases, the physical phenomenon becomes sophisticated and the presence of additional error sources is favorized.

The method under consideration here is developed by starting with the astrographic assumption of a restricted star field for which the equations (44) are valid. Hence, the dependences given by Equation (1), (2), and (3) can be deduced from the astrographic equations (44) as follows.

According to the least squares method, and starting with (44) as conditional equations, the normal equations result

$$\begin{aligned}(pxx)\,a + (pxy)\,b + (px)\,c &= (pxX)\\ (pxy)\,a + (pyy)\,b + (py)\,c &= (pyX)\\ (px)\;\;a + (py)\;\;b + (p)\;\;c &= (pX)\end{aligned} \tag{45}$$

which must be associated with

$$ax_0 + by_0 + c = X_0 \tag{46}$$

From (45) and (46) it results:

$$\begin{vmatrix} X_0 & x_0 & y_0 & 1\\ (pxX) & (pxx) & (pxy) & (px)\\ (pyX) & (pxy) & (pyy) & (py)\\ (pX) & (px) & (py) & (p)\end{vmatrix} = 0 \tag{47}$$

Since the same weights are assumed to affect both X and Y coordinates, the determinant (47) and the corresponding determinant for Y lead to

$$X_0 = D_1X_1 + D_2X_2 + \ldots D_nX_n \tag{48}$$

and

$$Y_0 = D_1Y_1 + D_2Y_2 + \ldots D_nY_n \tag{49}$$

where $D_i (i=1, 2, 3, \ldots, n)$ are the generalized dependences given by

$$D_i = up_ix_i + vp_iy_i + wp_i \tag{50}$$

According to the method of dependences, the quations (48) and (49) are a logical development from

$$\begin{aligned}D_1x_1 + D_2x_2 + \ldots D_nx_n &= x_0\\ D_1y_1 + D_2y_2 + \ldots D_ny_n &= y_0\\ D_1 \;\;+ D_2 \;\;+ \ldots D_n \;\;&= 1\end{aligned} \tag{51}$$

After considering the condition

$$\sum D_i^2 = \text{minimum} \tag{52}$$

the expressions (51) and (52) lead to the least squares solution given by (4′). (See also GENARO and TAFFARA, 1946.)

Finally, for three reference stars ($n=3$), the equations (51) become (1), (2) and (3) which shows that the astrographic method is identical with the three stars dependence method. (See also SMART, 1944, page 404; DE KORT, 1955). For more information on the dependence method, the studies by AREND (1931, 1932, 1933) and PLUMMER (1932) are recommended.

In order to discuss the effect of errors of the star catalogue, let us consider differential formulas for three stars derived from (48) and (49):

$$dX_0 = D_1\,dX_1 + D_2\,dX_2 + D_3\,dX_3 \tag{53}$$

and

$$dY_0 = D_1\,dY_1 + D_2\,dY_2 + D_3\,dY_3 \tag{54}$$

Assuming that the dX_i and dY_i in (53) and (54) represent only random errors, we logically have

$$\sigma_{X_0}^2 = D_1^2\sigma_{X_1}^2 + D_2^2\sigma_{X_2}^2 + D_3^2\sigma_{X_3}^2 \tag{55}$$

and

$$\sigma_{Y_0}^2 = D_1^2\sigma_{Y_1}^2 + D_2^2\sigma_{Y_2}^2 + D_3^2\sigma_{Y_3}^2 \tag{56}$$

where σ_{X_0}, σ_{Y_0}, σ_{X_1}, etc., denote standard deviations.

When the errors in right ascension are considered equal to those in declination, all standard deviations of the reference stars are equal, let us say, to σ.

Thus, (55) and (56) become

$$\sigma_{X_0}^2 = \sigma_{Y_0}^2 = \sigma^2(D_1^2 + D_2^2 + D_3^2) \tag{57}$$

The variances in X_0 and Y_0 are minimum if

$$D_1^2 + D_2^2 + D_3^2 = \text{minimum} \tag{58}$$

For defining the value of D_1, D_2, and D_3 which satisfy (58), let us consider the identity

$$(D_1 - D_2)^2 + (D_2 - D_3)^2 + (D_3 - D_1)^2 = 3(D_1^2 + D_2^2 + D_2^2) - (D_1 + D_2 + D_3)^2 \tag{59}$$

Taking into account (3), the identity (59) can be written as

$$D_1^2 + D_2^2 + D_3^2 = \tfrac{1}{3}[(D_1 - D_2)^2 + (D_2 - D_3)^2 + (D_3 - D_1)^2 + 1] \tag{60}$$

Evidently, (60) is a minimum when

$$D_1 = D_2 = D_3 = \tfrac{1}{3} \tag{61}$$

and (57) becomes

$$\sigma_{X_0} = \sigma_{Y_0} = \pm\frac{\sigma}{\sqrt{3}} \tag{62}$$

In other words, the effect of the errors arising from the star catalogue is a minimum when the flare occupies the baricenter of the three reference stars. This fact shall be taken into account by selecting the reference stars which enter in this reduction method.

As for the effect of the plate reading error in the computed flare position, it should be minimized when again the flare occupies the centroid of well-placed reference stars (ideally an equilateral triangle) in such a restricted field that the linear astrographic solution is justified (PLUMMER 1932 and DE KORT, 1955).

The question now arises: Are more than three stars needed for the best performance of this method and for the most accurate computation of the astronomic flare location? From the above discussion upon the minimum effect of the error propagation and in view of the existing astronomic works on dependences, the answer is quite simple. Obviously, the general solution of the dependence method is always possible (see Formula (4′)). However, it is hard to believe that more than three stars do improve the final result significantly enough to justify the additional work. On the contrary, more stars imply a more extensive and sophisticated sky field which opposes the restricted field of the classical astrographic method. When one discusses the dependence method discovered by SCHLESINGER (1926) and improved by COMRIE (1929), now generally adopted in astronomy, one has in mind the three star dependence method.

The remarkable simplifications introduced by the method of dependences, without losing accuracy, have been pointed out by COMRIE (1929) and other astronomers. These qualities are fully utilized in the present triangulation problem. Here, even more than in astronomy, the minimum number of three reference stars for the astronomical location of the flare is desirable. One should not be misled into believing that redundant star observations would improve the flare location, for one must recognize that in geodesy the stars do not furnish the final position of the object. The stars only assist in the computation of the conditional equations (39) and (40) which later enter in the adjustment of the triangulation. On the contrary, the redundancy of stars observations necessarily increase the number of the v-terms in the observation equations (39) and (40) without any improvement in the adjusted space location of the flare, e.g., the proper unknowns X, Y and Z. According to the least squares foundation, a higher accuracy for the adjusted flare location is expected by increasing the number of the observation equations (not of the reference stars). In other words, we have to be aware of a multi-camera station method with a minimum star observation.

Since stars, comets and asteroids are considered at infinity, the visual rays from different astronomical camera stations are parallel and the astronomers do not have linear measurements and their related problems. Hence, the location of astronomic objects is always expressed in spherical (angular) coordinates, e.g., right ascension α and declination δ.

The Geodesy, however, deals very much with lengths, because the geodetic objects are always at a finite distance. Here the rays connecting the various camera stations and the flare intersect and lead to additional geometric properties expressed by the conditional equations of intersection. Obviously, these equations are not ideal because of the errors in the plate readings and of the linearizing process. Nevertheless, they are proper for the adjustment of the triangulation and their deduction from only three stars is optimum.

On the line of these thoughts, some quantitative estimates are helpful:

Experience shows that the three stars dependence method in conjunction with a 800 mm focus camera lead to a standard deviation of the flare position of about ± 1.7 arc seconds.

Since each flare position will be computed independently from all available triangles of stars, it is assumed that at least two independent computations of this kind will be possible and the accuracy will increase to about $\pm 1.7/\sqrt{2}$.

Furthermore, the direction of the ray joining the camera station and the flare is given by two equations; the accuracy of the flare location is then expressed by $\pm 1.7/\sqrt{2\times 2}$.

In this triangulation method of multi-camera stations, a minimum set of three cameras should operate simultaneously. (For improving the accuracy more cameras are needed.) Thus, a standard deviation expressed by $\pm 1.7/\sqrt{2\times 2\times 3}$ is expected.

Finally, a rigorous block adjustment of the triangulation is expected to result in an additional 15% improvement; and the overall accuracy of the stellar triangulation is given by a standard deviation $\sigma \approx \pm 0.42$ arc seconds.

It is to be pointed out that this remarkable accuracy is higher than the astronomic accuracy (± 0.70 arc seconds) of the star catalogues used in the computation. This illustrates the benefit of the aforementioned conditional equations of intersection which are familiar in geodesy while missing in astronomy.

Undoubtedly the most important feature of the dependence method is the computation of the spherical coordinates (α_0, δ_0) of the flare directly from the observations without the need for forming standard coordinates.

This makes possible the development of Equations (39) and (40) in terms of plate readings in the same way as the development of theodolite equations in terms of circular readings are done. In other words, the transition from the well-known triangulation with fixed targets to the stellar triangulation with mobile targets is a matter of replacing the theodolite by the photographic camera while the structure of the mathematical model remains unchanged.

Since the photographic observations, unlike the theodolite observations, are free of the systematic reference to gravity, the stellar triangulation represents the highest degree of accuracy ever obtained in geodesy.

To advance the error analysis associated with Equations (9) and (10) it is necessary to show that the dependences D_1, D_2 and D_3 are invariant to any linear transformation of coordinates which shows that the knowledge of the center of the plate expressed in spherical coordinates ($\bar{\alpha}$, $\bar{\delta}$) is immaterial as far as the computation of the flare coordinates (α_0, δ_0) is concerned.

Obviously, the equations (1), (2) and (3) lead to

$$D_1(x_1 - x_0) + D_2(x_2 - x_0) + D_3(x_3 - x_0) = 0 \tag{63}$$

and

$$D_1(y_1 - y_0) + D_2(y_2 - y_0) + D_3(y_3 - y_0) = 0. \tag{64}$$

Let us now consider the general transformation of axes:

$$\begin{aligned} x_i' &= a + bx_i + cy_i \\ y_i' &= d + ex_i + fy_i \end{aligned} \qquad (i = 1, 2, 3). \tag{65}$$

Replacing (65) in (63) and (64) results in:

$$\sum_{i=1}^{3} D_i[b(x_i - x_0) + c(c_i - y_0)] = 0 \tag{66}$$

and

$$\sum_{i=1}^{3} D_1 \left[e(x_i - x_0) + f(y_i - y_0) \right] = 0 \tag{67}$$

or

$$b \sum_{i=1}^{3} D_i (x_i - x_0) + c \sum_{i=1}^{3} D_i (y_i - y_0) = 0 \tag{68}$$

and

$$e \sum_{i=1}^{3} D_i (x_i - x_0) + f \sum_{i=1}^{3} D_i (y_i - y_0) = 0 . \tag{69}$$

In view of (63) and (64), the equations (68) and (69) are identically satisfied which proves the invariance of $D_i (i = 1, 2, 3)$ to any linear transformation of coordinates. Thus, the center of the plate is immaterial in the computation of the flare location.

5. Passive and Active Satellites

There are two types of targets: (1) passive or sun-illuminated balloon satellites and (2) active or self-illuminated satellites provided with a flashing light source.

(1) The most important advantage of the passive satellites is their reliable and inexpensive sunlight source.

However, to convert the passive satellite into a target for stellar triangulation, it is necessary to chop its photographic trails in such a way that each set of images corresponds to a satellite position in space.

It is difficult to meet this goal even with the most accurate camera accessories. It happens that the attempt of exact synchronization of the shutter openings has no practical value here because of the inequal distance between the satellite and the various camera stations (YEAGER, 1964).

To overcome this difficulty, SCHMID, (1964) suggested the computation of the location of the satellite image by the help of a forced fitting of the observations into a 5th order time polynomial. In other words, the true physical satellite location has been replaced by a ficticious, computed location; and this is what one can do with passive satellite observations.

(2) The active satellite eliminates any operational difficulty and computational deficiency of this kind: The registration of the flashing light is punctual and it makes the timing and chopping of the trail unnecessary; the simultaneity condition of observation is automatically satisfied, the unique space location of the point of intersection of the rays is materialized and thus no computation of a ficticious satellite is needed.

On the other hand, a reliable package of flashing lights, powerful enough to be visible from more than one thousand kilometers, might be a severe engineering problem.

Apparently, this problem has been solved – after years of experiments – at the Patrick Air Force Base in Florida, where the launching of missiles provided with hundreds of flashing lights and photographed from distant camera stations, should presently be a routine matter.

In conclusion, it is felt that the observation of the passive satellite which is in itself a dependable and inexpensive target, requires difficult field operations while the active satellite which is rather difficult to build asks for a less expensive camera system and for simplified field and office operations.

6. Stationary and Nonstationary Camera Systems

The reader has been informed throughout this report about two camera systems which might be used in the stellar triangulation: (1) the stationary or photogrammetric camera and (2) the nonstationary or guided camera.

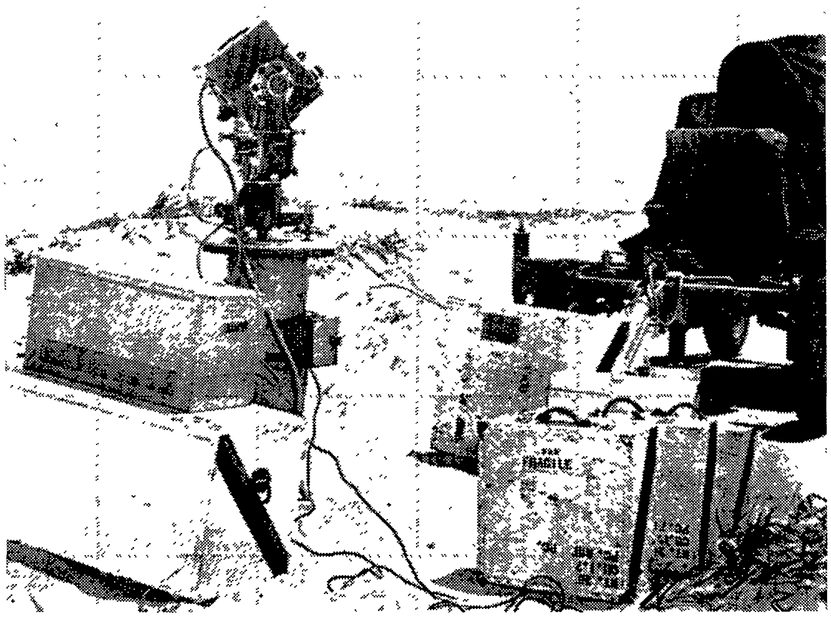

Fig. 6. Ballistic camera site with steel pier.

A variety of both types of cameras have been presented and described by VEIS (1963). Since these cameras have been designed for quite different purposes, e.g. ballistic trajectories, orbital and velocity satellite tracking, star statistics, none of them fully satisfies the specific needs of a mobile stellar triangulation project. Certain cameras are too heavy to be used in a geodetic field operation, others are of poor construction, others are rather sophisticated while provided with improper telescopes (short-focus), etc.

However, it is expected that the guided camera of the Geodetic Institute in Helsinki, which is presently in development under Prof. Väisälä's supervision, will meet the requirements of the stellar triangulation.

The stationary or photogrammetric system has been conceived with the ballistic camera Wild BC-4 (300 mm focal length) in mind. Some further features of this system are pointed out below:

Due to the earth rotation, the earth-fixed camera leads to the familiar trail imagery of the stars. The trails, in turn, have to be chopped in order to convert them into punctual images proper to comparator readings. The chopping operation requires costly accessories which in themselves mean additional error sources and complicate the overall system. Since each star is represented by more than one image point, the star identification becomes cumbersome and sometimes problematic.

In addition, it is to say that the relatively poor star registration of the photogrammetric system makes it improper for the data reduction approach presented in this report; most of the faint stars required by the dependency method do not register on even the most sensitive emulsions when the light source is moving on the fixed camera plate.

Fig. 7. Ballistic camera site with concrete pier.

A higher magnitude star registration is only possible with the guided camera which compensates the earth rotation. Its extreme reliability has been proven for more than one century. All existing astrographic catalogues are the product of the guided camera observations.

The photogrammetric data reduction method, suggests the splitting of the computation in two independent parts: (1) the adjustment of camera orientation-calibration, and (2) the adjustment of the triangulation (BROWN, 1957). The camera orientation-calibration computed before-hand is assumed to be free of error and unchanged during the event; its value is then introduced into the succeeding adjustment of the triangulation.

This is the typical forced adjustment *(Zwangsausgleichung)* largely discussed by many geodesists.

Of course, the block adjustment *(die Ausgleichung in einem Guß)* which is free of assumptions (BOLTZ, 1923, 1939) should be preferred.

When the camera is equatorially mounted and driven (guided) while the target is an active satellite, the block adjustment becomes quite easy. Its mathematical model is expressed by Equations (19) and (20), or (39) and (40).

Similar simplifications show the following mathematical model of the stellar triangulation suggested by VÄISÄLÄ and OTERMA (1960):

$$\cos\delta_0 \cos\beta_i \Delta\bar{t} + \sin\beta_i \Delta\bar{\delta} - p_i = 0 \qquad (70)$$

where $\Delta\bar{t}$ and $\Delta\bar{\delta}$ are corrections to some assumed hour angle and declination of one camera station, while another camera is considered as origin.

This approach is referring likewise to the guided camera system and uses the same

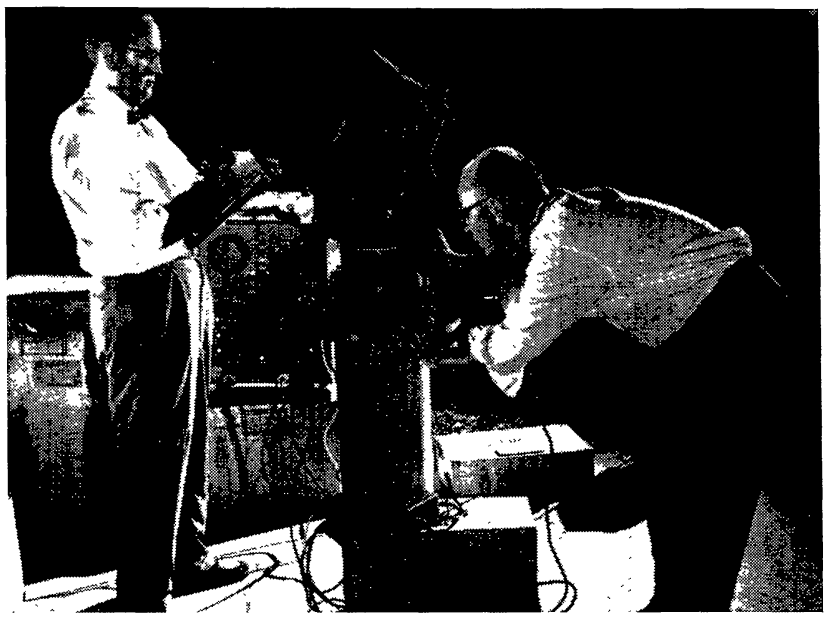

Fig. 8. Setting up the BC–4 camera on steel pedestal.

three star dependence method for computing the direction cosines to the target. However, the aim of Equation (70) is to compute the orientation between two non-intervisible camera stations while Equations (39) and (40) deal with the coordinates of targets and of unknown camera stations.

It is felt that a conditional equation of type (70) could be obtained from (39) and (40) after the elimination of $\mathrm{d}X, \mathrm{d}Y, \mathrm{d}Z$. The elimination process becomes straightforward in the case of the Hiran Triangulation (CORPACIUS, 1960).

The reliability of the photogrammetric system depends essentially from the physical stability of the camera during the field operation and from the theoretical and practical validity of the method of star orientation and calibration.

To satisfy somehow the camera stability, the pre- end post-star calibrations are to be quick operations made shortly before and after the photography of the satellite. For minimizing environmental disturbances, the camera is held in a temperature controlled dome (Figure 2) and supported on an isolated and stable pedestal (Figures

6, 7 and 8). Before the observations, the temperature in the dome is brought to the outside conditions to allow for thermal stabilization, etc.

The field operations with a guided camera are greatly simplified. Here the star images are punctual and thus no synchronization and gearing devices are needed. When an active satellite provided with short duration flashing lights is used, the camera operation is reduced to a single opening and closing of the shutter.

As for the accuracy of the observations, it is found difficult to believe that anyone could secure better results with a short-focus photogrammetric camera than with a longer focus guided telescope if the quality of the optics is comparable (HERGET, 1962).

Nevertheless, a world-wide stellar triangulation remains the all time major and most complex geodetic enterprise. Besides the international cooperation among nations, the project requires an extensive planning made by scientists who are familiar with international geodesy and accomplished by engineers who know the endurance and frustrations of the geodetic field work (SWANSON, 1964).

Acknowledgments

The author is particularly indebted to LTV Temco Aerosystems, without whose assistance the final revision for publication of this report would not have been possible.

The theoretical section of this report was sent for analysis and criticism to various selected scientists.

A great appreciation is expressed for the encouraging comments received from Prof. Dr. H. BODEMÜLLER, President of the German Geodetic Commission; Prof. Gino CASSINIS, late President of the Institute of Technology, Milan, Italy, and former President of the International Association of Geodesy; Prof. Dr. E. GIGAS, Director of the Institute for Applied Geodesy, Germany; Prof. Dr. W. HOFMANN, Dean of the Engineering School and Director of the Geodetic Institute, Bonn University, Germany; Brig. M. HOTINE, late Director of Overseas Surveys, England, presently associated with U. S. Coast and Geodetic Survey, Washington, D. C.; Prof. Dr. M. KNEISSL, Bavarian Academy of Sciences, Director of the German Geodetic Research Institute; Prof. Dr. R. ROELOFS, Director of the Geodetic Laboratory of the Institute of Technology, Delft, Holland; Prof. Dr. H. WOLF, Director of the Institute for Theoretic Geodesy, Bonn University, Germany.

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